NATURAL CONVECTION IN A TURBULENT BOUNDARY LAYER AT A WALL WITH VARIABLE TEMPERATURE

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The problem of natural convection in a turbulent boundary layer at a wall with variable temperature (or thermal flux) is solved by the integrals method.

The equations of a steady-state boundary layer at a wall whose radius of curvature may be disregarded relative to the thickness of this boundary layer can be written as

$$uu_x + vu_y = \beta g\theta + \frac{1}{\rho}\tau_y;$$
$$u\theta_x + v\theta_y = \frac{1}{c\rho}q_y; \quad (1)$$
$$u_x + v_y = 0.$$

Here $\theta = T - T_\infty$.

The boundary conditions are

$$y = 0;\quad q = q_w(x) \text{ or } \theta = \theta_w(x);\quad u = v = 0;\quad y = \delta;\quad u = 0;\quad \theta = 0. \quad (2)$$

For laminar flow with $\theta_W = x^n$ or $q_W = x^n$ this problem has been analyzed in [3, 4]. For turbulent flow this problem has been analyzed with the assumption that θ_W = const [2], q_W = const [1, 5], or q_W = const and $g \sim x$ [5].

Integrating Eq. (1) with respect to y from 0 to δ yields [2]

$$\frac{d}{dx}\int_0^\delta u^2 dy = \beta g \int_0^\delta \theta dy - \frac{1}{\rho}\tau_w;$$
$$\frac{d}{dx}\int_0^\delta u\theta dy = \frac{1}{\rho c}q_w. \quad (3)$$

It is assumed here that the thicknesses of the dynamic and of the thermal boundary layer are of the same order of magnitude, i.e., that Pr ~ 1.

We will stipulate the velocity profile u and the temperature profile θ as follows:

$$u = u_1 P_1(\eta);\quad \theta = \theta_w P_2(\eta);\quad \eta = \frac{y}{\delta}. \quad (4)$$

Inserting (4) into (3), we have

$$\alpha_1 \frac{d}{dx}(\delta u_1^2) = \alpha_2 \beta g \delta \theta_w - \frac{\tau_w}{\rho};$$

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$$\alpha_3 \frac{d}{dx}(\delta\theta_w u) = \frac{q_w}{\rho c}, \tag{5}$$

where

$$\alpha_1 = \int_0^1 P_1 d\eta; \quad \alpha_2 = \int_0^1 P_2 d\eta; \quad \alpha_3 = \int_0^1 P_1 P_2 d\eta. \tag{6}$$

Equations (5) can be solved numerically if $\theta_W(x)$, $q_W(x)$, $\tau_W(x)$, and the initial values of u_1 and δ are known. An analytical solution is possible for several forms of the $\theta_W(x)$ function.

We will seek the solution to (5) in the form

$$u_1 = u_0 z^k; \quad \delta = \delta_0 z^m; \quad \theta_w = \theta_0 z^p; \quad g = g_0 z^t, \tag{7}$$

where $z \equiv (1 + x - x_0)$ for a boundary layer with a nonzero initial thickness or $z \equiv x$ for a boundary layer with a zero initial thickness. Then expressions (5) yield

$$\begin{gathered} \alpha_1\delta_0 u_0^2 (m+2k)\, z^{m+2k-1} = \alpha_2 \beta g_0 \delta_0 \theta_0 z^{m+p+t} - \frac{\tau_w}{\rho}; \\ \alpha_3 \delta_0 \theta_0 u_0 (m+k+p)\, z^{m+k+p-1} = \frac{q_w}{\rho c}. \end{gathered} \tag{8}$$

In order to close system (8), it is necessary to add a relation for τ_W and q_W. Let us express τ_W as in [2]:

$$\tau_w = B\rho u_1^2 \left(\frac{\nu}{u_1 \delta}\right)^s, \tag{9}$$

with coefficient B and exponent s determined experimentally. In order to find the exponents k, m, and p, it is necessary to stipulate q_W as follows:

$$q_w = q_0 z^r, \tag{10}$$

or to use the Reynolds analogy [2]

$$\frac{q_w}{\tau_w} = \frac{c\theta_w}{u_1}. \tag{11}$$

Both methods yield the same result:

$$\begin{gathered} k = \frac{(1+2s)+(1+s)t}{3+2s} + \frac{1+s}{3+2s} r; \\ p = \frac{2s-t-1}{3+2s} + \frac{2+2s}{3+2s} r; \\ m = \frac{3-2s-ts}{3+2s} - \frac{s}{3+2s} r. \end{gathered} \tag{12}$$

With the exponent r of the thermal flux function known, one can find the exponents κ, p, and m of functions u_1, θ_W, and δ, respectively. If the θ_W function is given, then Eqs. (12) can be rewritten in terms of p instead.

We will write (11) as in [2]:

$$\frac{q_w}{\rho c} = \frac{\tau_w}{\rho} \cdot \frac{\theta_w}{u_1} = B \,\mathrm{Pr}^{-\frac{2}{3}} \nu^s u_0^{1-s} \delta_0^{-s} \theta_0 \, z^{p-ks-ms+k} \tag{13}$$

and then insert (9) and (13) into system (8) so that for z = 1 we obtain

$$\begin{gathered} \alpha_1 \alpha_4 \delta_0 u_0^2 = \alpha_2 \beta g_0 \delta_0 \theta_0 - B\nu^s u_0^{2-s} \delta_0^{-s}; \\ \alpha_3 \alpha_5 \delta_0 u_0 = B \,\mathrm{Pr}^{-\frac{2}{3}} \nu^s \delta_0^{-s} u_0^{1-s}. \end{gathered} \tag{14}$$

Here $\alpha_4 = m + 2k$ and $\alpha_5 = m + k + p$.

If the solution to Eq. (5) is sought in the form

$$u_1 = u_0 \exp(kz); \quad \delta = \delta_0 \exp(mz); \quad \theta_w = \theta_0 \exp(pz);$$
$$g = g_0 \exp(tz); \quad q_w = q_0 \exp(rz), \tag{15}$$

then we obtain formulas analogous to (12);

$$k = \frac{1+s}{3+2s} r + \frac{1+s}{3+2s} t;$$
$$p = \frac{2+2s}{3+2s} r - \frac{t}{3+2s}; \tag{16}$$
$$m = \frac{-sr}{3+2s} - \frac{st}{3+2s}.$$

In this case, for determining δ_0, u_0, θ_0, and q_0 we have a system which becomes identical to (14) at $z = 0$.

Solving system (14), we find δ_0, u_0, and q_0 as functions of θ_0:

$$\delta_0 = a_1 \Pr^{\frac{-2}{3+3s}} (\mathrm{Gr})_{z=1}^{\frac{-s}{2+2s}} \left(1 + a_2 \Pr^{\frac{2}{3}}\right)^{\frac{s}{2+2s}};$$
$$u_0 = a_3 \nu (\mathrm{Gr})_{z=1}^{\frac{1}{2}} \left(1 + a_2 \Pr^{\frac{2}{3}}\right)^{-\frac{1}{2}}; \tag{17}$$
$$q_0 = a_4 \lambda \Pr^{\frac{1+3s}{3+3s}} (\mathrm{Gr})_{z=1}^{\frac{1}{2+2s}} \left(1 + a_2 \Pr^{\frac{2}{3}}\right)^{\frac{-1}{2+2s}},$$

where the constant coefficients a_1, a_2, a_3, and a_4 are

$$a_1 = \left(\frac{B}{\alpha_3 (m+k+p)}\right)^{\frac{1}{1+s}} \left(\frac{\alpha_1 (m+2k)}{\alpha_2}\right)^{\frac{s}{2+2s}};$$
$$a_2 = \frac{\alpha_3 (m+k+p)}{\alpha_1 (m+2k)};$$
$$a_3 = \left(\frac{\alpha_1 (m+2k)}{\alpha_2}\right)^{-\frac{1}{2}}; \tag{18}$$
$$a_4 = B^{\frac{1}{1+s}} \left(\frac{1}{\alpha_3 (m+k+p)}\right)^{\frac{-s}{1+s}} \left(\frac{\alpha_1 (m+2k)}{\alpha_2}\right)^{-\frac{1}{2+2s}}.$$

If the thermal flux q_0 is given, then we obtain analogous formulas:

$$\delta_0 = c_1 \Pr^{\frac{s-2}{3+2s}} (\mathrm{Gr}^*)_{z=1}^{-\frac{s}{3+2s}} \left(1 + a_2 \Pr^{\frac{2}{3}}\right)^{\frac{s}{3+2s}};$$
$$u_0 = c_2 \Pr^{-\frac{1+3s}{3(3+2s)}} \nu (\mathrm{Gr}^*)_{z=1}^{\frac{1+s}{3+2s}} \left(1 + a_2 \Pr^{\frac{2}{3}}\right)^{-\frac{1+s}{3+2s}}; \tag{19}$$
$$\theta_0 = c_3 \Pr^{-\frac{2+6s}{3(3+2s)}} q_0 \lambda (\mathrm{Gr}^*)_{z=1}^{-\frac{1}{3+2s}} \left(1 + a_2 \Pr^{\frac{2}{3}}\right)^{\frac{1}{3+2s}}.$$

The constants c_1, c_2, and c_3 are

$$c_1 = B^{\frac{3}{3+2s}} \left(\frac{\alpha_1 (m+2k)}{\alpha_2}\right)^{\frac{s}{3+2s}} \left(\frac{1}{\alpha_3 (m+k+p)}\right)^{\frac{3-s}{3+2s}};$$
$$c_2 = B^{-\frac{1}{3+2s}} \left(\frac{\alpha_1 (m+2k)}{\alpha_2}\right)^{-\frac{1+s}{3+2s}} \left(\frac{1}{\alpha_3 (m+k+p)}\right)^{\frac{s}{3+2s}}; \tag{20}$$
$$c_3 = B^{-\frac{2}{3+2s}} \left(\frac{\alpha_1 (m+2k)}{\alpha_2}\right)^{\frac{1}{3+2s}} \left(\frac{1}{\alpha_3 (m+k+p)}\right)^{\frac{2s}{3+2s}}.$$

The heat-transfer coefficient is defined as follows:

$$\alpha = \frac{q_w}{\theta_w}; \quad \mathrm{Nu}_z = \frac{\alpha z}{\lambda}.$$

If the wall temperature θ_W is given, then we obtain

$$\mathrm{Nu}_z = a_4 \mathrm{Pr}^{\frac{1+3s}{3+2s}} (\mathrm{Gr})_{z=1}^{\frac{1}{2+2s}} \left(1 + a_2 \mathrm{Pr}^{\frac{2}{3}}\right)^{\frac{-1}{2+2s}} \theta_0^{-1} z^{r-p+1}. \tag{21}$$

If the thermal flux at the wall q_W is given, then we obtain

$$\mathrm{Nu}_z = c_3^{-1} \mathrm{Pr}^{\frac{2+6s}{9+6s}} \lambda^{-2} (\mathrm{Gr}^*)_{z=1}^{\frac{1}{3+2s}} \left(1 + a_2 \mathrm{Pr}^{\frac{2}{3}}\right)^{\frac{-1}{3+2s}} z^{r-p+1}. \tag{22}$$

Let us now consider specific cases. The profiles of velocity and temperature in a turbulent boundary layer are usually defined in the form:

$$P_1 = \eta^{\alpha}(1-\eta)^{\beta}; \quad P_2 = 1 - \eta^{\gamma}. \tag{23}$$

The values $\alpha = 1/7$ and $\beta = 4$ given in [2] agree closely with the experiment where θ_W = const. The values $\alpha = 1/10$, $\beta = 4$, and $\gamma = 1/10$ given in [7] have been obtained experimentally with q_W = const. Depending on the value assumed for α, one obtains different values for s in the Blasius formula [2]:

$$s = \frac{2\alpha}{1+\alpha}. \tag{24}$$

If g = const (t = 0), θ_W = const (p = 0), B = 0.0225, $\alpha = 1/7$, $\beta = 4$, $\gamma = 1/7$, and s = 1/4, then

$$m = \frac{7}{10}; \quad k = \frac{1}{2}; \quad r = \frac{1}{5}. \tag{25}$$

The solution (17) with (25) is obviously identical to that in [2] for a constant wall temperature.

If g = const (t = 0), q_W = const (r = 0), B = 0.0225, $\alpha = 1/7$, $\beta = 4$, $\gamma = 1/7$, and s = 1/4, then

$$m = \frac{5}{7}; \quad k = \frac{3}{7}; \quad p = -\frac{1}{7}. \tag{26}$$

The solution (19) with (26) is obviously identical to that in [1] for a constant thermal flux. If g ~ x and q_W = const, then (12) and (19) yield the solution given in [5].

It follows from (12) that $p \geq 0$ if

$$r \geqslant \frac{1+t-2s}{2+2s}. \tag{27}$$

If condition (27) is not satisfied, then $\theta_W \to \infty$ at $x \to 0$ and, therefore, such solutions apply to a boundary layer with a positive initial thickness. After comparing this solution with the test results in [6, 7] for θ_W = const and q_W = const, one may conclude that they agree when

$$\begin{aligned} \mathrm{Ra} &\geqslant 10^{13} \quad \text{at} \quad \theta_w = \text{const}; \\ \mathrm{Ra}^* &\geqslant 10^{15} \quad \text{at} \quad q_w = \text{const}, \end{aligned} \tag{28}$$

where Ra = Gr Pr; Ra* = Gr* Pr. Such values of the Rayleigh number correspond to a fully developed turbulent flow mode.

Obviously, for lower values of the Rayleigh number the procedure can be modified, and for a small x one must apply solutions which have been obtained for laminar flow, with this procedure then extended beyond a certain point where (28) is true according to the solution for a turbulent boundary layer of a nonzero initial thickness. The structure of our solution here indicates that formulas (12) are based on the assumption of (7), (9), and (11) or (7), (9), and (10) being valid. If there exists a solution in a power or exponential form, therefore, then a sufficient condition for the feasibility of closing system (5) and obtaining the values of coefficients u_0, δ_0, and $\theta_0(q_0)$ in (7) is that the Reynolds relation (11) hold true for at least one point. The solution obtained here is also applicable to values of the Rayleigh number below those stipulated in (28) but, in order that it agree with experiments, parameters s and B in the Blasius formula (9) must be determined according to the procedure in [8].

An analogous solution can be obtained for a boundary layer with compound convection.

NOTATION

x, y are the coordinates referred to the generatrix of the body surface;
u, v are the velocity components along x and y, respectively;
ρ is the density;
c is the specific heat at constant pressure;
ν is the kinematic viscosity;
θ_w is the temperature at the wall;
q_w is the thermal flux at the wall;
τ_w is the friction stress at the wall;
g is the acceleration of gravity;
β is the thermal expansivity;
Pr is the Prandtl number;
Gr is the Grashof number;
Ra is the Rayleigh number;
Nu is the Nusselt number.

LITERATURE CITED

1. Progress in Heat Transmission [in Russian], Pt. 4, Moscow (1970).
2. E. R. Eckert and R. M. Dreick, Theory of Heat and Mass Transfer [Russian translation], Moscow (1961).
3. M. Tribus, Trans. ASME, 80C, 1180 (1958).
4. E. M. Sparrow and J. L. Gregg, Trans. ASME, 80C, No. 2 (1958).
5. Lemlich and Verdi, Trans. ASME, Ser. C, No. 4 (1964).
6. T. Fujii, Internat. J. Heat and Mass Transfer, 13, No. 5 (1970).
7. Vliet, Trans. ASME, Ser. C, No. 4 (1969).
8. P. M. Brdlik, Inzh.-Fiz. Zh., 13, No. 2 (1967).
9. V. I. Bronshtein, T. L. Perel'man, and B. N. Yudaev, in: Heat and Mass Transfer [in Russian], Vol. 1, Pt. 1, Minsk (1972).